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MIXED SUMS OF SQUARES AND TRIANGULAR NUMBERS (III)

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Abstract. In this paper we confirm a conjecture of Sun which states that each positive integer is a sum of a square, an odd square and a triangular number. Given any positive integer m, we show that $p = 2m + 1$ is a prime congruent to 3 modulo 4 if and only if $T_m = m(m + 1)/2$ cannot be expressed as a sum of two odd squares and a triangular number, i.e., $p^2 = x^2 + 8(y^2 + z^2)$ for no odd integers x, y, z. We also show that a positive integer cannot be written as a sum of an odd square and two triangular numbers if and only if it is of the form $2T_m$ $(m > 0)$ with $2m + 1$ having no prime divisor congruent to 3 modulo 4.

1. Introduction

The study of expressing natural numbers as sums of squares has long history. Here are some well-known classical results in number theory.

(a) (Fermat-Euler theorem) Any prime $p \equiv 1 \pmod 4$ is a sum of two squares of integers.

(b) (Gauss-Legendre theorem, cf. [G, pp. 38–49] or [N, pp. 17–23]) $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as a sum of three squares of integers if and only if n is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$.

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(c) (Lagrange's theorem) Every $n \in \mathbb{N}$ is a sum of four squares of integers.

Those integers $T_x = x(x+1)/2$ with $x \in \mathbb{Z}$ are called triangular numbers. Note that $T_x = T_{-x-1}$ and $8T_x + 1 = (2x+1)^2$. In 1638 P. Fermat asserted that each $n \in \mathbb{N}$ can be written as a sum of three triangular numbers (equivalently, $8n+3$ is a sum of three squares of odd integers); this follows from the Gauss-Legendre theorem.

Let $n \in \mathbb{N}$. As observed by L. Euler (cf. [D, p. 11]), the fact that $8n+1$ is a sum of three squares (of integers) implies that n can be expressed as a sum of two squares and a triangular number. This is remarkable since there are infinitely many natural numbers which cannot be written as a sum of three squares. According to [D, p. 24], E. Lionnet stated, and V. A. Lebesgue [L] and M. S. Réalis [R] showed that n is also a sum of two triangular numbers and a square. In 2006 these two results were re-proved by H. M. Farkas [F] via the theory of theta functions. Further refinements of these results are summarized in the following theorem.

Theorem 1.0. (i) (B. W. Jones and G. Pall [JP]) *For every $n \in \mathbb{N}$, we can write $8n+1$ in the form $8x^2 + 32y^2 + z^2$ with $x, y, z \in \mathbb{Z}$, i.e., n is a sum of a square, an even square and a triangular number.*

(ii) (Z. W. Sun [S07]) *Any natural number is a sum of an even square and two triangular numbers. If $n \in \mathbb{N}$ and $n \neq 2T_m$ for any $m \in \mathbb{N}$, then n is also a sum of an odd square and two triangular numbers.*

(iii) (Z. W. Sun [S07]) *A positive integer is a sum of an odd square, an even square and a triangular number unless it is a triangular number T_m $(m > 0)$ for which all prime divisors of $2m+1$ are congruent to 1 mod 4.*

We mention that Jones and Pall [JP] used the theory of ternary quadratic forms and Sun [S07] employed some identities on q-series. Motivated by Theorem 1.0(iii) and the fact that every prime $p \equiv 1 \pmod 4$ is a sum of an odd square and an even square, the second author [S09] conjectured that each natural number $n \neq 216$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}$, where p is a prime or zero. Sun [S09] also made a general conjecture which states that for any $a, b \in \mathbb{N}$ and $r = 1, 3, 5, \ldots$ all sufficiently large integers can be written in the form $2^a p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r mod 2^b.

In [S07] Sun investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$) represent all natural numbers, and left two conjectures in this direction. In [GPS] S. Guo, H. Pan and Sun proved Conjecture 2 of [S07]. Conjecture 1 of Sun [S07] states that any positive integer n is a sum of a square, an odd square and a triangular number, i.e., $n - 1 = x^2 + 8T_y + T_z$ for some $x, y, z \in \mathbb{Z}$.

In this paper we prove Conjecture 1 of Sun [S07] and some other results concerning mixed sums of squares and triangular numbers. Our main

result is as follows.

Theorem 1.1. (i) *Each positive integer is a sum of a square, an odd square and a triangular number. A triangular number T_m with $m \in \mathbb{Z}^+$ is a sum of two odd squares and a triangular number if and only if $2m+1$ is not a prime congruent to 3 mod 4.*

(ii) *A positive integer cannot be written as a sum of an odd square and two triangular numbers if and only if it is of the form $2T_m$ ($m \in \mathbb{Z}^+$) with $2m+1$ having no prime divisor congruent to 3 mod 4.*

Remark 1.1. In [S09] the second author conjectured that if a positive integer is not a triangular number then it can be written as a sum of two odd squares and a triangular number unless it is among the following 25 exceptions:

$$4,\ 7,\ 9,\ 14,\ 22,\ 42,\ 43,\ 48,\ 52,\ 67,\ 69,\ 72,\ 87,\ 114,$$
$$144,\ 157,\ 159,\ 169,\ 357,\ 402,\ 489,\ 507,\ 939,\ 952,\ 1029.$$

Here is a consequence of Theorem 1.1.

Corollary 1.1. (i) *An odd integer $p > 1$ is a prime congruent to 3 mod 4 if and only if $p^2 = x^2 + 8(y^2 + z^2)$ for no odd integers x, y, z.*

(ii) *Let $n > 1$ be an odd integer. Then all prime divisors of n are congruent to 1 mod 4, if and only if $n^2 = x^2 + 4(y^2 + z^2)$ for no odd integers x, y, z.*

Remark 1.2. In number theory there are very few simple characterizations of primes such as Wilson's theorem. Corollary 1.1(i) provides a surprising new criterion for primes congruent to 3 mod 4.

In the next section we will prove an auxiliary theorem. Section 3 is devoted to our proofs of Theorem 1.1 and Corollary 1.1.

2. An auxiliary theorem

In this section we prove the following auxiliary result.

Theorem 2.1. *Let m be a positive integer.*

(i) *Assume that $p = 2m+1$ be a prime congruent to 3 mod 4. Then T_m cannot be written in the form $x^2 + y^2 + T_z$ with $x, y, z \in \mathbb{Z}$, $x^2 + y^2 > 0$ and $x \equiv y \pmod 2$. Also, $2T_m$ is not a sum of a positive even square and two triangular numbers.*

(ii) *Suppose that all prime divisors of $2m+1$ are congruent to 1 mod 4. Then T_m cannot be written as a sum of an odd square, an even square and a triangular number. Also, $2T_m$ is not a sum of an odd square and two triangular numbers.*

To prove Theorem 2.1 we need the following result due to Hurwitz.

Lemma 2.1 (cf. [D, p. 271] or [S07, Lemma 3]). *Let n be a positive odd integer, and let $p_1, \ldots, p_r$ be all the distinct prime divisors of n congruent to 3 mod 4. Write $n = n_0 \prod_{0<i\leqslant r} p_i^{\alpha_i}$, where $n_0, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$ and n_0 has no prime divisors congruent to 3 mod 4. Then*

$$|\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = n^2\}| = 6n_0 \prod_{0<i\leqslant r} \left(p_i^{\alpha_i} + 2\frac{p_i^{\alpha_i} - 1}{p_i - 1}\right).$$

As in [S07], for $n \in \mathbb{N}$ we define

$$r_0(n) = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + T_y + T_z = n \text{ and } 2 \mid x\}|$$

and

$$r_1(n) = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + T_y + T_z = n \text{ and } 2 \nmid x\}|.$$

By p. 108 and Lemma 2 of Sun [S07], we have the following lemma.

Lemma 2.2 ([S07]). *For $n \in \mathbb{N}$ we have*

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = n \text{ and } x \equiv y \pmod 2\}| = r_0(2n)$$

and

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = n \text{ and } x \not\equiv y \pmod 2\}| = r_1(2n).$$

Also, $r_0(2T_m) - r_1(2T_m) = (-1)^m(2m+1)$ for every $m \in \mathbb{N}$.

Proof of Theorem 2.1. By Lemma 2.2,

$$\begin{aligned}
&r_0(2T_m) + r_1(2T_m)\\
=&|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = T_m\}|\\
=&\frac{1}{2}|\{(x,y,z) \in \mathbb{Z}^3 : 8x^2 + 8y^2 + (8T_z + 1) = 8T_m + 1\}|\\
=&\frac{1}{2}|\{(x,y,z) \in \mathbb{Z}^3 : 4(x+y)^2 + 4(x-y)^2 + (2z+1)^2 = (2m+1)^2\}|\\
=&\frac{1}{2}|\{(u,v,z) \in \mathbb{Z}^3 : 4(u^2+v^2) + (2z+1)^2 = (2m+1)^2\}|\\
=&\frac{1}{6}|\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = (2m+1)^2\}|.
\end{aligned}$$

(i) As $p = 2m+1$ is a prime congruent to 3 mod 4, by Lemma 2.1 and the above we have

$$r_0(2T_m) + r_1(2T_m) = p + 2.$$

On the other hand,

$$r_0(2T_m) - r_1(2T_m) = (-1)^m(2m+1) = -p$$

by Lemma 2.2. So

$$2r_0(2T_m) = r_0(2T_m) + r_1(2T_m) + (r_0(2T_m) - r_1(2T_m)) = p + 2 - p = 2.$$

Therefore

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = T_m \text{ and } 2 \mid x - y\}| = r_0(2T_m) = 1$$

and also

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + T_y + T_z = 2T_m \text{ and } 2 \mid x\}| = r_0(2T_m) = 1.$$

Since $T_m = 0^2 + 0^2 + T_m$ and $2T_m = 0^2 + T_m + T_m$, the desired results follow immediately.

(ii) As all prime divisors of $2m+1$ are congruent to 1 mod 4, we have

$$r_0(2T_m)+r_1(2T_m) = \frac{1}{6}|\{(x,y,z) \in \mathbb{Z}^3 : x^2+y^2+z^2 = (2m+1)^2\}| = 2m+1$$

in view of Lemma 2.1. Note that m is even since $2m+1 \equiv 1 \pmod 4$. By Lemma 2.2, $r_0(2T_m) - r_1(2T_m) = (-1)^m(2m+1) = 2m+1$. Therefore

$$|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + T_z = T_m \text{ and } 2 \nmid x - y\}| = r_1(2T_m) = 0.$$

This proves part (ii) of Theorem 2.1. □

3. Proofs of Theorem 1.1 and Corollary 1.1

Lemma 3.1. *Let $m \in \mathbb{N}$ with $2m+1 = k(w^2+x^2+y^2+z^2)$ where $k, w, x, y, z \in \mathbb{Z}$. Then*

$$2T_m = k^2(wy+xz)^2 + k^2(wz-xy)^2 + 2T_v \quad \textit{for some } v \in \mathbb{Z}.$$

Proof. Write the odd integer $k(w^2+x^2-(y^2+z^2))$ in the form $2v+1$. Then

$$\begin{aligned} 8T_m + 1 =& (2m+1)^2 = k^2(w^2+x^2+y^2+z^2)^2 \\ =& (2v+1)^2 + 4k^2(w^2+x^2)(y^2+z^2) \\ =& 8T_v + 1 + 4k^2((wy+xz)^2 + (wz-xy)^2) \end{aligned}$$

and hence

$$2T_m = 2T_v + k^2(wy + xz)^2 + k^2(wz - xy)^2.$$

This concludes the proof. $\square$

Proof of Theorem 1.1. (i) In view of Theorem 1.0(iii), it suffices to show the second assertion in part (i).

Let m be any positive integer. By Theorem 2.1(i), if $2m+1$ is a prime congruent to 3 mod 4 then T_m cannot be written as a sum of two odd squares and a triangular number.

Now assume that $2m+1$ is not a prime congruent to 3 mod 4. Since the product of two integers congruent to 3 mod 4 is congruent to 1 mod 4, we can write $2m+1$ in the form $k(4n+1)$ with $k, n \in \mathbb{Z}^+$.

Set $w = 1 + (-1)^n$. Observe that $4n + 1 - w^2$ is a positive integer congruent to 5 mod 8. By the Gauss-Legendre theorem on sums of three squares, there are integers x, y, z with x odd such that $4n + 1 - w^2 = x^2 + y^2 + z^2$. Clearly both y and z are even. As $y^2 + z^2 \equiv 4 \pmod 8$, we have $y_0 \not\equiv z_0 \pmod 2$ where $y_0 = y/2$ and $z_0 = z/2$.

Since $2m + 1 = k(w^2 + x^2 + y^2 + z^2)$, by Lemma 3.1 there is an integer v such that

$$2T_m = 2T_v + k^2(wy + xz)^2 + k^2(wz - xy)^2.$$

Thus

$$\begin{aligned} T_m =& T_v + 2(kwy_0 + kxz_0)^2 + 2(kwz_0 - kxy_0)^2 \\ =& T_v + (kwy_0 + kxz_0 + (kwz_0 - kxy_0))^2 \\ &+ (kwy_0 + kxz_0 - (kwz_0 - kxy_0))^2. \end{aligned}$$

As w is even and $kxy_0 \equiv y_0 \not\equiv z_0 \equiv kxz_0 \pmod 2$, we have

$$kwy_0 + kxz_0 \pm (kwz_0 - kxy_0) \equiv 1 \pmod 2.$$

Therefore $T_m - T_v$ is a sum of two odd squares.

(ii) In view of Theorem 1.0(ii) and Theorem 2.1(ii), it suffices to show that if $2m + 1$ ($m \in \mathbb{Z}^+$) has a prime divisor congruent to 3 mod 4 then $2T_m$ is a sum of an odd square and two triangular numbers.

Suppose that $2m+1 = k(4n-1)$ with $k, n \in \mathbb{Z}^+$. Write $w = 1+(-1)^n$. Then $4n-1-w^2$ is a positive integer congruent to 3 mod 8. By the Gauss-Legendre theorem on sums of three squares, there are integers x, y, z such that $4n - 1 - w^2 = x^2 + y^2 + z^2$. Clearly $x \equiv y \equiv z \equiv 1 \pmod 2$ and $2m + 1 = k(w^2 + x^2 + y^2 + z^2)$. By Lemma 3.1, for some $v \in \mathbb{Z}$ we have

$$2T_m = k^2(wy + xz)^2 + k^2(wz - xy)^2 + 2T_v.$$

Let $u = kwz - kxy$. Then

$$T_{v+u} + T_{v-u} = \frac{(v+u)^2 + (v-u)^2 + (v+u) + (v-u)}{2} = u^2 + 2T_v.$$

Thus

$$2T_m = (kwy + kxz)^2 + T_{v+u} + T_{v-u}.$$

Note that $kwy + kxz$ is odd since w is even and k, x, z are odd.

Combining the above we have completed the proof of Theorem 1.1. $\square$

Proof of Corollary 1.1. (i) Let $m = (p-1)/2$. Observe that

$$\begin{aligned} &T_m = T_x + (2y+1)^2 + (2z+1)^2 \\ \iff &p^2 = 8T_m + 1 = (2x+1)^2 + 8(2y+1)^2 + 8(2z+1)^2. \end{aligned}$$

So the desired result follows from Theorem 1.1(i).

(ii) Let $m = (n-1)/2$. Clearly

$$\begin{aligned} &2T_m = T_x + T_y + (2z+1)^2 \\ \iff &2n^2 = 16T_m + 2 = (2x+1)^2 + (2y+1)^2 + 8(2z+1)^2 \\ \iff &n^2 = (x+y+1)^2 + (x-y)^2 + 4(2z+1)^2. \end{aligned}$$

So $2T_m$ is a sum of an odd square and two triangular numbers if and only if $n^2 = x^2 + (2y)^2 + 4z^2$ for some odd integers x, y, z. (If x and z are odd but y is even, then $x^2 + (2y)^2 + 4z^2 \equiv 5 \not\equiv n^2 \pmod 8$.) Combining this with Theorem 1.1(ii) we obtain the desired result. $\square$

Remark 3.1. We can deduce Corollary 1.1 in another way by using some known results (cf. [E], [EHH] and [SP]) in the theory of ternary quadratic forms, but this approach involves many sophisticated concepts.

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